Erratum

The Horizontal No Free Lunch theorem takes $\tilde{T}$, the set of possible targets to be a partition of the search space Ω. However, in the case of multiple queries, they are treated as a single query to a larger search space Ω_Q. Any distribution of targets on the original search space will produce overlapping targets on Ω_Q. This prevents considering the set of possible targets as a partition of the search space Ω_Q. As such the horizontal no free lunch theorem does not properly handle multiple queries.

Theorem 1. *Given a uniform distribution over targets of cardinality k, and baseline uniform distribution, the average active information will be non-positive*

Proof. The average active information formula in its general form is

$$E[I_+] = E\left[-\log \frac{\phi(T)}{\psi(T)}\right] = E\left[\log \frac{\psi(T)}{\phi(T)}\right] \tag{1}$$

where T the random variable taking the domain of the powerset of Ω, $\phi(T)$ is the fixed probability of the baseline uniform search algorithm succeeding given a specific target and $\psi(T)$ is the fixed probability of the search algorithm being considered succeeding given a specific target.

By Jensen's inequality

$$E\left[\log \frac{\psi(T)}{\phi(T)}\right] \leq \log E\left[\frac{\psi(T)}{\phi(T)}\right] \tag{2}$$

Any given target can be represented by a function producing 1 for every point ω in the target, and 0 otherwise. All functions with the same number of targets will be closed under permutation. The algorithms succeed if they pick a point in the target, which is equivalent running single query and getting a 1. The No Free Lunch theorem holds for any distribution which is closed under permutation, and thus all algorithms will have on average, the same probability of success.

$$E[\psi(T)] = E[\phi(T)] = \frac{k}{|\Omega|} \tag{3}$$

We note that $\phi(T)$ is a constant, performing the same regardless of the identity of the target. It is a baseline uniform search, and thus will select every target with the same probability. Therefore,

$$E\left[\frac{\psi(T)}{\phi(T)}\right] = \frac{E[\psi(T)]}{\phi(T)} = \frac{\phi(T)}{\phi(T)} = 1 \tag{4}$$

This effectively restates the No Free Lunch theorem, the expected performance of two search algorithms does not differ. Using equation 2:

$$E[I_+] = E\left[\log \frac{\psi(T)}{\phi(T)}\right] \leq \log E\left[\frac{\psi(T)}{\phi(T)}\right] = \log \frac{E[\psi(T)]}{\phi(T)} = \log 1 = 0 \tag{5}$$

Therefore the expected active information is always non-positive.

Additionally, equality only occours if

$$\frac{\psi(T)}{\phi(T)} \tag{6}$$

is the same for all values of T. This will only occour if both algorithms succeed and fail with the same probability for every target.

□

Corollary 1. *Given a distribution over all targets, such that targets of the same cardinality have the same probability: the average active information will be non-positive*

$$E\left[\log \frac{\psi(T)}{\phi(T)}\right] \leq 0 \tag{7}$$

Proof. We can express the average active information as

$$E\left[\log\frac{\psi(T)}{\phi(T)}\right]=\sum_{k=0}^{k=|\Omega|}\Pr\left[|T|=k\right]E\left[\log\frac{\psi(T)}{\phi(T)}\middle||T|=k\right] \tag{8}$$

Since the probabilities for sets of the same size are the same, Theorem 1 applies to the inner expected value. The expression then takes the weighted average of non-positive values and thus will be non-positive. □

Theorem 2. *For every distribution of targets there is some baseline search for which the active information will always be non-positive.*

$$E[I_+]=E\left[-\log\frac{\phi(T)}{\psi(T)}\right]\leq 0 \tag{9}$$

for some ϕ*.*

Proof. We can rewrite the average active information as:

$$E[I_+]=E\left[-\log\frac{\phi(T)}{\psi(T)}\right]=E\left[-\log\phi(T)\right]-E\left[-\log\psi(T)\right] \tag{10}$$

For every possible distribution over targets, there exists an algorithm, ϕ, that maximizes $E\left[-\log\phi(T)\right]$. $E\left[-\log\phi(T)\right]\geq E\left[-\log\psi(T)\right]$ since ϕ maximizes the expectation, and thus, $E\left[-\log\phi(T)\right]-E\left[-\log\psi(T)\right]\leq 0$. □

The Horizontal No Free Lunch Theorem does hold in all cases; however, determining the baseline may be non-trivial.

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